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Introduction

If, in studying the scattering of fast electrons, one uses foils which are sufficiently thin, then, for all angles above a certain small critical angle,[1] multiple scattering may be supposed to be negligible. Furthermore, the number which is scattered inelastically is small[2] except at very small angles. That is, elastic single scattering is usually of primary importance.

The elastic scattering by light elements has been the subject of theoretical investigation by Mott[3–7] and others, and a general approximate formula has been developed. (Any possible effects due to shielding of the nucleus by the outside electrons have been neglected, i. e. the external electron is pictured as moving in a Coulomb field.) In this formula, it is supposed that $\alpha = (Z/137)$ is small. This is not so for heavy elements, with the result that no simple closed formulae have been found. Mott[8] has derived an exact expression for the scattering of a Dirac electron in a Coulomb field, as a series in Legendre (associated) polynomials. It was the purpose of the present research to sum this series numerically for a typical heavy element (mercury). The results should be approximately true for the nearby elements (gold, lead, thallium, and bismuth).

[1] See, for instance, W. A. Fowler, Phys. Rev. 54, 773 (1938) for further references.

[2] H. Bethe, Handbuch der Physik, Bd XXIV/1, zw. Aufl, S. 501.

[3] N. F. Mott, Proc. Roy. Soc. (A) 124, 425 (1929).

[4] O. Scherzer, Ann. d. Phys., 13, 137 (1932).

[5] F. Sauter, Z. f. Phys., 86, 818 (1933).

[6] J. Meixner, Z. f. Phys., 90, 312 (1934).

[7] W. H. Furry, Phys. Rev., 46, 391 (1934).

[8] N. F. Mott, Proc. Roy. Soc. (A) 135, 429 (1932). It should be noted that there are misprints in formulae (3.19), (4.2), and (4.4). These formulae are given correctly in the earlier article, ref. (3) above. On page 445, $A' = (e^{3i\beta-(3/2)\pi i}) \times A$, and formula (5.26) should have θ instead of β, while Ψ_c should be similar to Ψ_a. Also, p. 451, $R = (q/2)\{\Gamma(1-iq)/\Gamma(1+iq)\}e^{2iq \log \sin \theta/2}$.

General Formulae

According to Mott, the intensity of scattering is $ff^* + gg^*$, where

$$\begin{aligned} f &= \tfrac{1}{2}\int_0^\infty y^{-1-iq}\, e^{-y}\, \Phi_b(y)dy \\ g &= \tfrac{1}{2}\int_0^\infty y^{-1-iq}\, e^{-y}\, \Phi_d(y)dy \end{aligned} \tag{1}$$

$$\Phi_b(y) = i\sum_{k=0}^{\infty}[k(k-iq')c_k y^{\rho_k} + (k+1)(-k-1-iq')c_{k+1}y^{\rho_{k+1}}](-)^k P_k(\cos\theta)$$

$$\Phi_d(y) = i\sum_{k=1}^{\infty}[(k-iq')c_k y^{\rho_k} + (k+iq'+1)c_{k+1}\, y^{\rho_{k+1}}](-)^k P_k^1(\cos\theta)$$

$\rho = (k^2-\alpha^2)^{1/2}$, $q = \alpha/\beta$, $\beta = v/c$, $q' = q(1-\beta^2)^{1/2}$, and $c_k = -e^{-i\pi\rho}/\Gamma(\rho + iq + 1)$.

Now abbreviate by letting $S_\rho = \dfrac{e^{-\pi i\rho}}{\rho + iq}\dfrac{\Gamma(\rho - iq)}{\Gamma(\rho + iq)}$,

and note that $\int_0^\infty y^{-1-iq+\rho}e^{-y}\,dy = \Gamma(\rho - iq)$. Also, let $D_k = S_k - S_\rho$. Integrating (1) term by term, we have

$$\left.\begin{aligned} f &= -\tfrac{1}{2}i\sum_{k=0}^{\infty}\{k(k-iq')\,S_{\rho_k} - (k+1)(k+1+iq')\,S_{\rho_{k+1}}\}(-)^k P_k(\cos\theta) \\ &= -iq'F + G \\ &\text{and} \\ g &= -\tfrac{1}{2}i\sum_{k=1}^{\infty}\{(k-iq')\,S_{\rho_k} + (k+iq'+1)\,S_{\rho_{k+1}}\}(-)^k P_k^1(\cos\theta) \\ &= -iq'H + K \end{aligned}\right\} \tag{2}$$

$$F = -\tfrac{1}{2}i\sum[kS_{\rho_k} + (k+1)\,S_{\rho_{k+1}}](-)^k P_k$$

$$G = -\tfrac{1}{2}i\sum[k^2 S_{\rho_k} - (k+1)^2\,S_{\rho_{k+1}}](-)^k P_k$$

$$H = -\tfrac{1}{2}i\sum[S_{\rho_k} - S_{\rho_{k+1}}](-)^k P_k^1$$

$$K = -\tfrac{1}{2}i\sum[kS_{\rho_k} + (k+1)\,S_{\rho_{k+1}}](-)^k P_k^1$$

It will be proved in the Appendix that

$$g = [iq'(1+\cos\theta)F + (1-\cos\theta)G]/\sin\theta \tag{2a}$$

It then follows that

$$ff^* + gg^* = (q'^2FF^*/\sin^2 \tfrac{1}{2}\theta) + (GG^*/\cos^2 \tfrac{1}{2}\theta) \quad (2b)$$

Mott uses a subscript zero to denote values when $\alpha = 0$. He then sets $F_1 = F - F_0$, and $G_1 = G - G_0$, and obtains

$$\left.\begin{aligned} F_0 &= \frac{i}{2}\left(\sin\frac{\theta}{2}\right)^{2iq} \frac{\Gamma(1-iq)}{\Gamma(1+iq)} = iR/q \\ F_1 &= \tfrac{1}{2}\, i\, \Sigma[kD_k + (k+1)D_{k+1}]\,(-)^k P_k \\ G_0 &= R\cot^2\frac{\theta}{2} \\ G_1 &= \tfrac{1}{2}\, i\, \Sigma[k^2D_k - (k+1)^2D_{k+1}](-)^kP_k \end{aligned}\right\} \quad (3)$$

We shall show (see Appendix) that

$$k^2D_k = -(\pi i\alpha^2/2)kS_k + (1/8)[\pi^2\alpha^4 - 4\alpha^2(1+2iq)]\,S_k + E_k \quad (4)$$

where $E_k = O(1/k^2)$ for large k.

Since S_k is of order $(1/k)$, D_k will then be of order $(1/k^2)$. Also, approximately, $kD_k + (k+1)D_{k+1} = -(\pi i\alpha^2/2)(S_k + S_{k+1}) = -(\pi i\alpha^2/2)S_k\,(2iq+1)/(k+iq+1) = O(1/k^2)$.

Therefore, F_1 converges rapidly, and need cause no further concern. This is not true of G_1.

$$G_1 = \tfrac{1}{2}i(-\pi i\alpha^2/2)\sum_{k=0}^{\infty}[kS_k - (k+1)S_{k+1}]\,(-)^kP_k + \tfrac{1}{2}i(1/8)\,[\pi^2\alpha^4 - 4\alpha^2(1+2iq)]\sum_{k=0}^{\infty}(S_k - S_{k+1})(-)^kP_k + \tfrac{1}{2}i\sum_{k=0}^{\infty}(E_k - E_{k+1})\,(-)^kP_k = G_1{}^{\Phi} + G_1{}^{F_0} + G_1{}^{E}$$

The first series has been summed by Mott, and a similar procedure may be used for the second series (see Appendix). The third series converges rapidly, so that we need only find the first few terms (numerically).

$$G_1 = [(\pi\alpha^2/4)(1+\cos\theta)\Phi/\Gamma(1+2iq)] + (iF_0/8q)\,[\pi^2\alpha^4 - 4\alpha^2(1+2iq)] + (i/2)\sum(E_k - E_{k+1})\,(-)^kP_k \quad (5)$$

In formula (5)

$$\Phi = \left(2\sin\frac{\theta}{2}\right)^{2iq-2}\int_0^1\left(\frac{1-z}{z}\right)^{iq}\left(1 + z\cot^2\frac{\theta}{2}\right)^{-\frac{1}{2}}dz$$

Substituting $y = 1 - z$,

$$\Phi = \left(2\sin\frac{\theta}{2}\right)^{2iq-2}\sin\frac{\theta}{2}\int_0^1\left(\frac{y}{1-y}\right)^{iq}\left(1 - y\cos^2\frac{\theta}{2}\right)^{-\frac{1}{2}}dy$$
$$= \left(2\sin\frac{\theta}{2}\right)^{2iq-2}\sin\frac{\theta}{2}\,\Gamma(iq+1)\;\Gamma(iq-1)\; F\left(\tfrac{1}{2}, iq+1, 2; \cos^2\frac{\theta}{2}\right) \quad (6)$$

where F is the hypergeometric function.[9, 10] The series converges well for $\theta \geq 60°$, but for smaller angles becomes impractical. In this region, it is best to use the connection formula[11]

$$\Gamma\left(\frac{3}{2}\right)\Gamma(-iq+1)\;\Gamma(\tfrac{1}{2})\;\Gamma(iq+1)\; F\left(\tfrac{1}{2}, iq+1, 2; \cos^2\frac{\theta}{2}\right)$$
$$= \Gamma(\tfrac{1}{2})\,\Gamma(iq+1)\,\Gamma(-iq+\tfrac{1}{2})\; F\left(\tfrac{1}{2}, iq+1, iq+\tfrac{1}{2}; \sin^2\frac{\theta}{2}\right) \quad (7)$$
$$+ \Gamma\left(\frac{3}{2}\right)\Gamma(-iq+1)\;\Gamma(iq-\tfrac{1}{2})\left(\sin^2\frac{\theta}{2}\right)^{-iq+\frac{1}{2}} F\left(\frac{3}{2}, -iq+1, -iq+\frac{3}{2}; \sin^2\frac{\theta}{2}\right)$$

Scattering at Small Angles

At high velocities, $q' \to 0$, and hence the main contribution to the scattering comes from the G term. If we retain only that part of G_1 which has a singularity at $\theta = 0$ (i.e. $G_1{}^{\Phi}$), then

$$G_1 \cong (\pi\alpha^2/4)\,(1+\cos\theta)\,(2\sin\tfrac{1}{2}\theta)^{2iq-2}\sin\tfrac{1}{2}\theta\;|\Gamma(iq+1)|^2\frac{F_c}{\Gamma(1+2iq)}$$
$$\cong (\pi\alpha^2/8)\sin\tfrac{1}{2}\theta\,(2\sin\tfrac{1}{2}\theta)^{2iq}\,\Gamma(\tfrac{1}{2}-iq)\;\Gamma(1+iq)/\Gamma\left(\frac{3}{2}\right)\Gamma(1+2iq)$$

by substituting (7) and neglecting the second right-hand member.

(We have put $F_c \equiv F(\tfrac{1}{2}, iq+1, 2, \cos^2\tfrac{1}{2}\theta)$.)

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[9] We are deeply indebted to Professor J. R. Oppenheimer for pointing out to one of us (JHB) that this integral is a hypergeometric function. This knowledge saved us considerable labor.

[10] Whittaker and Watson, Modern Analysis, 4th ed., p. 293.

[11] Whittaker and Watson, Modern Analysis, 4th ed., p. 291.

Now the duplication formula for $z = \frac{1}{2} + iq$ is

$$2^{2iq}\ \Gamma(\tfrac{1}{2}+iq)\ \Gamma(1+iq) = \pi^{1/2}\ \Gamma(1+2iq).$$

Using this, we find

$$G \cong \cot^2 \tfrac{1}{2}\theta\ \{-iq\,F_0 + (\pi\alpha^2/4)\ (\sin \tfrac{1}{2}\theta)^{2iq+1}\ [\Gamma(\tfrac{1}{2}-iq)/\Gamma(\tfrac{1}{2}+iq)]\}$$

which could be written in the form

$$G \cong e^{i\psi} \cot^2 \tfrac{1}{2}\theta\ \{(q/2) + (\pi\alpha^2/4) \sin \tfrac{1}{2}\theta\ e^{i\chi}\}$$

where χ and ψ are real angles and

$$e^{i\chi} = \Gamma(\tfrac{1}{2}-iq)\ \Gamma(1+iq)\ /\ \Gamma(\tfrac{1}{2}+iq)\ \Gamma(1-iq)$$

Then,

$$GG^* \cong \cot^4 \tfrac{1}{2}\theta[(q^2/4) + (\pi\alpha^2 q/4) \sin \tfrac{1}{2}\theta \cos \chi] \quad (8)$$

For $\alpha = 0$, and $q' = 0$ ($v = c$), this reduces[12] to Mott's formula (ref. 8, eq. 6.9). If q is small with respect to $\frac{1}{2}$, then χ is small, and $\cos \chi \cong 1$. We then have the improved formula already given by Mott (ref. 3, p. 438, equation before (25)). However, as Mott himself has remarked, this formula does not hold for gold (since $q_{min} = .58$) and the polarization effects cannot be calculated at all satisfactorily until we take in more terms. Any correction occasioned by the inclusion of $\cos \chi$ will reduce the value of the scattering below that found by Mott. For $q = .6$, we find $\chi = 1.10722$, and $\cos \chi = .44715$. Substituting in (8), we find that the scattering is not very different, at angles less than 15°, from that calculated with the exact formula. The deviations at larger angles are, of course, to be expected. (See Fig. 1 for the comparison.)

Polarization

According to Mott, the asymmetry to be expected in a double scattering (90°) experiment is measured by $2\delta = 2[(fg^* - gf^*)/(ff^* + gg^*)]^2$. Let $F = a + ib$, and $G = c + id$. Then $FG^* + GF^* = 2(ac + bd)$. $fg^* - gf^* = -iq' (\cot \frac{1}{2}\theta + \tan \frac{1}{2}\theta)$ $(FG^* + GF^*) = -4iq'\ (ac + bd)/\sin\theta = -4iq'$ $(ac + bd)$ for angles of 90°. The polarization thus goes to zero as $v \rightarrow c$, because $q' \rightarrow 0$.

Calculation Procedure

In order to find F_1 and G_1, one needs to know S_k, S_ρ, and the derived quantities D_k and E_k, as well as the hypergeometric functions in (6) and (7). The latter may be calculated simply by summing the series, while the former depend upon gamma functions of a complex argument.[13] We may note that the phases only of these gamma functions are needed for S_k and S_ρ. It is probably most accurate to make all the primary calculations of phases in the region where the asymptotic formula holds, and to extend inwards by repeatedly applying the formula $z\Gamma(z) = \Gamma(z+1)$. For large z, we have $\ln \Gamma(z) \sim (z - \frac{1}{2}) \ln z - z + \frac{1}{2} \ln 2\pi + (1/12\,z) - (1/360\,z^3) + \ldots\ldots$ Let $z = re^{i\theta} = a + ib$, and $\Gamma(z) = Re^{i\psi}$. Then $\ln R + i\psi \sim (a + ib - \frac{1}{2})(\ln r + i\theta) - a - ib + \frac{1}{2} \ln 2\pi + (a - ib)/(12r^2) - (a^3 - 3a^2bi - 3ab^2 + ib^3)/360r^6 + \ldots\ldots$. Equating imaginary parts,

$$\psi \sim (a - \tfrac{1}{2})\,\theta + b\,(\ln r - 1) - (b/12r^2) + (3a^2b - b^3)/(360r^6) + \ldots\ldots \quad (9)$$

If $b^2 << 3a^2$, then

$$\psi \sim (a - \tfrac{1}{2})\theta + b\,(\ln r - 1) - (b/12r^2) + (b/120r^4) + \ldots\ldots \quad (10)$$

Formula (10) is simpler to apply in most cases, but we have used formula (9) for $q \geq 1.5 (a = 5$ and $6)$. The calculation of S_ρ is made most conveniently in the polar co-ordinate system, where one needs only to add angles.

Once we have D_k, we may find E_k from formula (4), and then G_1 from (5). F_0, F_1, and G_0 are obtained by using formula (3), and the calculation of scattering intensity is completed with an application of (2b).

In spite of the complexity of our calculations, we are confident of their correctness for the following reasons: (1) They have been made in duplicate, i. e. by two men working independently, so that local errors have been eliminated; (2) Our results on polarization check with those of Mott as closely as would be expected, so that there are no systematic errors at 90°; and (3) our values of r at other angles are consistent with those at 90°, and give a family of curves varying smoothly with q.

Results

It is convenient to tabulate the ratio r of our calculated scattering intensity to ordinary Rutherford scattering. That is, we multiply our values by $(4/q^2) \sin^4 \frac{1}{2}\theta$. In Table I, we list the values of the ratio r, together with those of δ (polarization), as a function of q or of ε $(= E/mc^2)$.

In Figure 1, we have plotted (1) our exact values of r for all our values of q, (2) the values of r for $q = 0.6$ on the basis of our formula (8),

[12] One must, of course, use formula (2b) of this paper in making the comparison.

[13] The functions of an imaginary argument (modulus ≤ 1) have been tabulated by H. T. Davis, Tables of the Higher Mathematical Functions, vol. I, p. 272. An excellent idea of the general behavior of the functions of a complex argument may be gained by looking at the graph of Jahnke-Emde, Funktionentafeln, (1933), S. 90.

TABLE I

VALUES OF r AND δ

q	ϵ	15°	30°	45°	60°	90°	120°	150°	180°	δ
.6	3.35	1.12	1.34	1.62	1.85	1.89	1.30	.51	.15	.012
.65	1.28	1.09	1.29	1.55	1.78	1.89	1.44	.82	.54	.036
.73	.666	1.06	1.22	1.45	1.67	1.86	1.59	1.19	1.00	.057
.8	.463	1.04	1.17	1.37	1.58	1.82	1.68	1.43	1.31	.064
.9	.314	1.03	1.11	1.28	1.48	1.77	1.75	1.65	1.59	.068
1.0	.232	1.02	1.07	1.21	1.39	1.71	1.78	1.80	1.78	.066
1.2	.145	1.01	1.02	1.09	1.25	1.58	1.79	1.97	1.98	.057
1.5	.086					1.39	1.73	2.05	2.31	
2.0	.046					1.19	1.58	2.04	2.35	.027

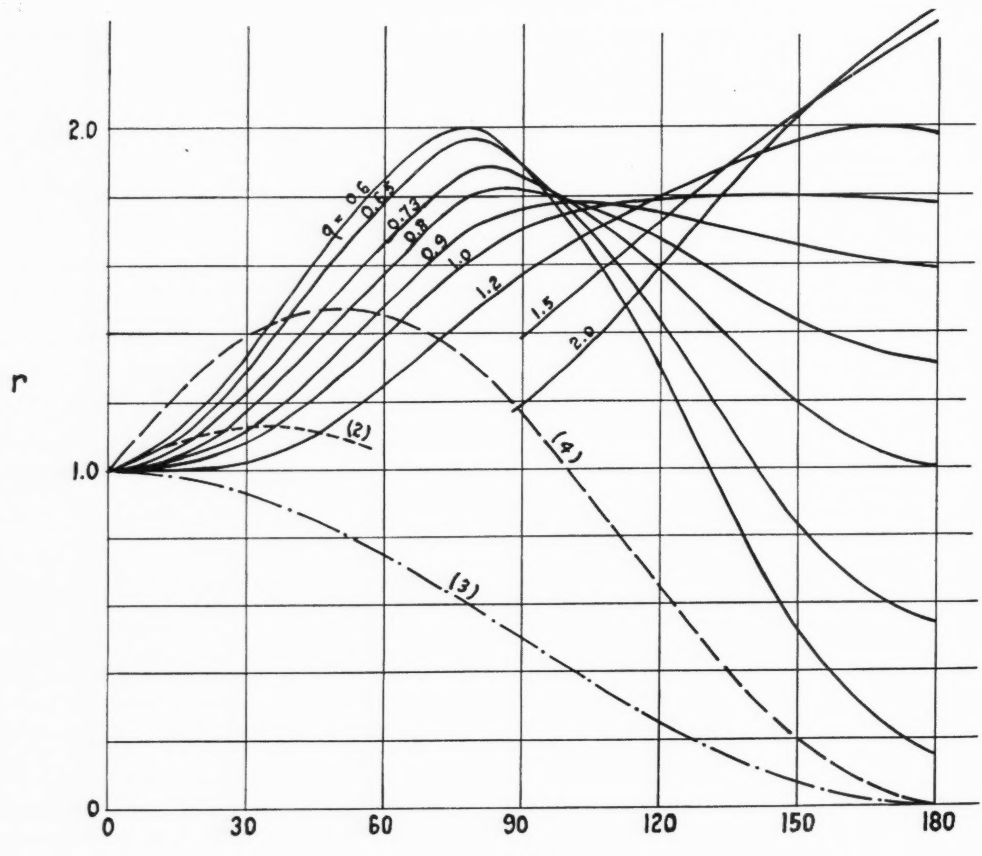

FIGURE 1. $\dfrac{\text{Calculated scattering}}{\text{"Rutherford" scattering}}$ vs. Angle.

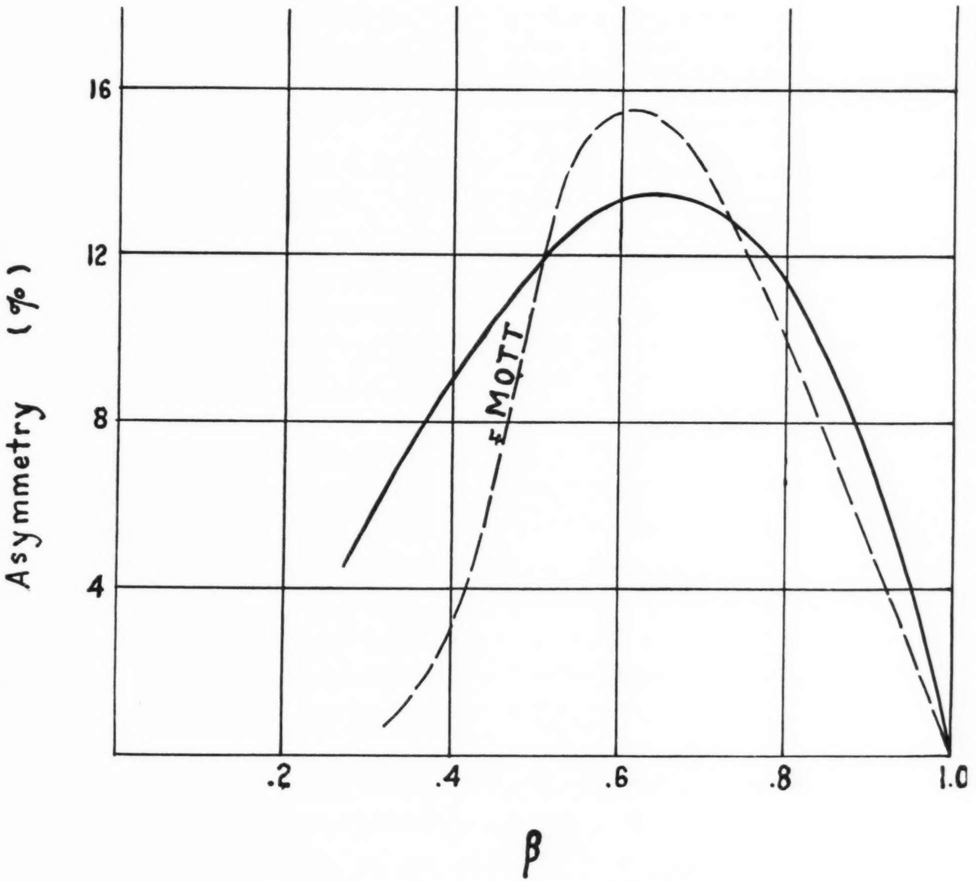

FIGURE 2. Asymmetry per 100 electrons vs. velocity.

(3) the values of r found from Mott's light element formula with neglect of α^2, and (4) the same for Hg with inclusion of α^2. As previoulsy remarked, (4) cannot be used legitimately for heavy elements. The difference between (1) and (3) is great, and may be thought of as due to the spin-orbit interaction. Just how this interaction enters is not immediately obvious, but a detailed analysis of our results might possibly lead to an interpretation in terms of a model.

In Figure 2, we have plotted 2δ, and show Mott's results for comparison. It is seen that our polarization values are only slightly less than his at the maximum.

Since the scattering is approximately Rutherford at small angles, we can offer no explanation of the results of Fowler and Oppenheimer[1] on multiple scattering. Our angular distribution agrees well with that observed by Barber and Champion,[14] the numbers in the angular ranges 20°–30°, 30°–60°, and 60°–180° being in the ratio 37 : 27 : 9, as against the experimental ratio 37 : 30 : 8. However, the disagreement between theory and experiment with respect to absolute intensities[14] still remains.

ACKNOWLEDGMENTS

We are grateful to Professor J. R. Oppenheimer not only for the specific help referred to in footnote 9, but also for illuminating discussion of the subject in general. Furthermore, it would not have been possible to carry out this work as

[14] A. Barber and F. C. Champion, Proc. Roy. Soc. A 168, 159 (1938).

completely as here presented, had it not been for the computational assistance rendered available (1) by a grant from the Graduate School and (2) by Professor F. W. Loomis. We wish to thank the men who aided in the computations, Messrs. F. N. Gillette, R. W. Potts (NYA), and P. R. Gillette, for their capable and efficient work.

Appendix

I. *Derivation of Equation (4)*

Expand S_ρ in a Taylor's series, as follows:

$$S_\rho = S_k + \left.\frac{dS_\rho}{d\rho}\right|_k (\rho - k) + \left.\frac{d^2S_\rho}{d\rho^2}\right|_k \frac{(\rho-k)^2}{2} + \ldots \quad \text{(A1)}$$

$$\rho = k\left(1 - \frac{\alpha^2}{k^2}\right)^{1/2} = k\left\{1 - \frac{\alpha^2}{2k^2} + \frac{3}{8}\frac{\alpha^4}{k^4}\ldots\ldots\right\} \quad \text{(A2)}$$

We shall consistently neglect terms of order (S_k/k^3), or of order $(1/k^4)$. Denote $\frac{d}{dz}\log\Gamma(z)$ by $\psi(z)$. Since $\log S_\rho = -\pi i\rho + \log\Gamma(\rho - iq) - \log\Gamma(\rho + iq + 1)$, we have on differentiating:

$$\frac{1}{S_\rho}\frac{dS_\rho}{d\rho} = -\pi i + \psi(\rho - iq) - \psi(\rho + iq + 1)$$

$$\frac{1}{S_\rho}\frac{d^2S_\rho}{d\rho^2} - \frac{1}{S_\rho^2}\left(\frac{dS_\rho}{d\rho}\right)^2 = \psi'(\rho - iq) - \psi'(\rho + iq + 1)$$

The $(1/k^3)$ term in $\rho - k$ may be neglected, so that the terms contributing to $(dS_\rho/d\rho)_k$ will be of order S_k and (S_k/k), respectively. Likewise, contributions to $(d^2S_\rho/d\rho^2)_k$ will come only from terms of order S_k. Now, for large k,[15]

$$\psi(k + iq) \sim \tfrac{1}{2}\log(k^2 + q^2) + i\tan^{-1}(q/k) - [\tfrac{1}{2}(k + iq)]\ldots.$$

$$\psi(k - iq) \sim \tfrac{1}{2}\log(k^2 + q^2) - i\tan^{-1}(q/k) - [\tfrac{1}{2}(k - iq)]\ldots.$$

$$\psi(k - iq) - \psi(k + iq) \sim -2i\tan^{-1}(q/k) \cong -2iq/k \text{ if } q << k$$

$$\psi(k + iq) - \psi(k + iq + 1) = -1/(k+iq)$$

Hence $$\left.\frac{dS_\rho}{d\rho}\right|_k \cong S_k[-\pi i - [(1 + 2iq)/k]]$$

and $$\left.\frac{d^2S_\rho}{d\rho^2}\right|_k \cong -S_k\pi^2$$

Substituting in (A1), using (A2),

$$k^2D_k \cong -(\pi i\alpha^2/2)kS_k + (1/8)[\pi^2\alpha^4 - 4\alpha^2(1 + 2iq)]S_k \quad (4)$$

[15] H. T. Davis, Tables of the Higher Mathematical Functions, vol. 1, p. 283.

II. *Proof of Equation (2a)*

This equation will be correct,[16] if we can show that

$$H = -(1+\cos\theta)F/\sin\theta \quad \text{(A3)}$$

$$K = (1-\cos\theta)G/\sin\theta \quad \text{(A4)}$$

(a) *Proof of (A3)*

Since $g = 0$ for the non-relativity case ($\alpha = 0$ and $q = q'$), we have $K_0 = iqH_0$. In general, $K = \sin\theta\,\{d/d(\cos\theta)\}F$. If we apply this to F_0, we have $K_0 = \sin\theta\, f(q)\,(d/d\cos\theta)\,(1-\cos\theta)^{iq} = -\sin\theta\, f(q) iq(1-\cos\theta)^{iq-1} = -iq\,\{\sin\theta/(1-\cos\theta)\}F_0$. Accordingly, $H_0 = -[\sin\theta/(1-\cos\theta)]F_0$. Can we now show that $H_1 = -[\sin\theta/(1-\cos\theta)]F_1$?

$$H_1 = (i/2)\sum_{k=1}^{\infty}(D_k - D_{k+1})(-)^kP_k^1$$

$$= (i/2)\sum_{k=1}^{N}(D_k - D_{k+1})(-)^kP_k^1 + \left(\frac{i}{2}\right)\sum_{N+1}^{\infty}(D_k - D_{k+1})(-)^kP_k^1$$

Now, $D_k - D_{k+1} \cong -(\pi i\alpha^2/2)\,[(S_k/k) - [S_{k+1}/(k+1)]] = O(1/k^2)$.

Since the terms are of this order, the sum will be of order $(1/N)$ and may be neglected.

$$H_1 \cong (i/2)\,[-D_1(P_1^1 + P_2^1) + \ldots.. + D_N(-)^N(P_{N-1}^1 + P_N^1) - D_{N+1}(-)^NP_N^1]$$

$$\cong (i/2)\sum_{k=1}^{N}(-)^kD_k(P_{k-1}^1 + P_k^1)$$

But
$$P_k^1 + P_{k-1}^1 = k(P_{k-1} - P_k)(1 + \cos\theta)/\sin\theta$$
$$P_k^1 - P_{k-1}^1 = k(P_{k-1} + P_k)(1 - \cos\theta)/\sin\theta$$

Hence $$H_1 \cong (i/2)\left[\sum_{k=1}^{N}(-)^kD_kk(P_{k-1} - P_k)\right](1 + \cos\theta)/\sin\theta$$

$$\cong (i/2)\,[(1 + \cos\theta)/\sin\theta]\,\{-D_1(P_0 - P_1) + 2D_2(P_1 - P_2) + \ldots.(-)^NND_N(P_{N-1} - P_N)\}$$

$$\cong -(i/2)\,[(1+\cos\theta)/\sin\theta]\,\{D_1P_0 - P_1(D_1 + 2D_2) \ldots\ldots + (-)^{N-1}P_{N-1}[(N-1)D_{N-1} - ND_N]\} - (i/2)\,[(1+\cos\theta)/\sin\theta]\,(-)^{N+1}P_NND_N$$

$$\cong -(i/2)\,[(1+\cos\theta)/\sin\theta]\sum_{k=0}^{N-1}\{kD_k + (k+1)D_{k+1}\}(-)^kP_k$$

[16] We have also calculated g directly, i. e., from equation (2), and have obtained approximately the same results as with equation (2a). The approximative character of these results arises from the neglect of series terms with $k \geq 4$.

But $F_1 = (i/2)\sum_{k=0}^{N-1}\{kD_k + (k+1)D_{k+1}\}\,(-)^kP_k +$

$$(i/2)\sum_{N}^{\infty}\{kD_k+(k+1)D_{k+1}\}\,(-)^kP_k$$

$kD_k + (k+1)D_{k+1} \cong -(\pi i\alpha^2/2)\,[S_k + S_{k+1}] = O(1/k^2)$, so that the sum of these terms may be neglected.

Therefore $H_1 = -[(1+\cos\theta)/\sin\theta]\,F_1$, which proves equation (A3).

(b) *Proof of (A4).*

$$K_0 = [-iq\sin\theta/(1-\cos\theta)]\,F_0 = -iq\cot\frac{\theta}{2}F_0$$

$$G_0 = -iq\cot^2\frac{\theta}{2}F_0 = \cot\frac{\theta}{2}K_0 = [\sin\theta/(1-\cos\theta)]\,K_0 \quad \text{(A5)}$$

Can we show that $K_1 = G_1\tan(\theta/2)$?

$$K_1 = \tfrac{1}{2}i\sum_{k=1}^{\infty}[kD_k + (k+1)D_{k+1}](-)^kP_k^1$$

$$= -\tfrac{1}{2}i(\pi i\alpha^2/2)\sum_{k=1}^{\infty}[S_k + S_{k+1}](-)^kP_k^1$$

$$+\tfrac{1}{2}i(1/8)[\pi^2\alpha^4-4\alpha^2(1+2iq)]\sum_{k=1}\left(\frac{S_k}{k}+\frac{S_{k+1}}{k+1}\right)(-)^kP_k^1$$

$$+\tfrac{1}{2}i\sum_{k=1}^{\infty}\left[\frac{E_k}{k}+\frac{E_{k+1}}{(k+1)}\right](-)^kP_k^1 = K_1'+K_1''+K_1'''$$

The first term is found to be (see Appendix IIc)

$$K_1' = (\pi\alpha^2/4)\,\Phi\sin\theta/\Gamma(1+2iq) \quad \text{(A6)}$$

There is a corresponding term in G_1, namely

$$G_1^{\Phi} = (\pi\alpha^2/4)\,(1+\cos\theta)\Phi/\Gamma(1+2iq) = K_1'\cot\frac{\theta}{2} \quad \text{(A7)}$$

Now, the coefficients of P_k^1 in $K_1''+K_1'''$ are of order $(1/k^2)$, and so we may make the same transformation as in IIa.

$$K_1''+K_1''' = \tfrac{1}{2}i\sum_{k=1}^{\infty}(H_k+H_{k+1})\,(-)^kP_k^1$$

$$= \tfrac{1}{2}i\sum_{k=0}^{\infty}[kH_k-(k+1)H_{k+1}](-)^kP_k\,(1-\cos\theta)/\sin\theta$$

$$= [(1-\cos\theta)/\sin\theta]\,(G_1^{F_0}+G_1^E) \quad \text{(A8)}$$

Combining (A5), (A7), and (A8), we have (A4). It should be noticed that the transformation is not made on the series K_1 itself, for the term N^2D_N is not negligible.

(c) *Derivation of (A6)*[17]

$$\sum_{k=1}^{\infty}[S_k+S_{k+1}]\,(-)^kP_k^1 = [1/\Gamma(1+2iq)]\int_0^1 (1-x)^{2iq}x^{-iq-1}\sum_{k=1}^{\infty}(x^k-x^{k+1})P_k^1dx$$

$$= [1/\Gamma(1+2iq)]\sin\theta\int_0^1\left(\frac{1-x}{x^{1/2}}\right)^{2iq}\left(\frac{1-x}{x}\right)\frac{d}{d\mu}\sum_{k=0}^{\infty}x^kP_kdx$$

$$= [1/\Gamma(1+2iq)]\sin\theta\int_0^1\left(\frac{1-x}{x^{1/2}}\right)^{2iq}(1-x)\,(1-2x+x^2)^{-3/2}dx$$

$$= \Phi\sin\theta/\Gamma(1+2iq)$$

III *Evaluation of* $G_1^{F_0}$

$$\sum_{k=0}^{\infty}(S_k-S_{k+1})\,(-)^kP_k$$

$$= [1/\Gamma(1+2iq)]\int_0^{\infty}(1-x)^{2iq}x^{-iq}\,(1+x)\sum_{k=0}^{\infty}x^{k-1}P_kdx$$

$$= [1/\Gamma(1+2iq)]\int_0^{\infty}(1-x)^{2iq-1}x^{-iq}dx[(1-x^2)\,(1-2\mu x+x^2)^{-1/2}/x]$$

$$= [1/\Gamma(1+2iq)]\int_0^{\infty}(1-x)^{2iq-1}x^{-iq}dx\int^{\mu}\mathrm{F}(x,\mu)d\mu$$

$$= [2i\Gamma(2iq)/\Gamma(1+2iq)]\int^{\mu}f(\mu)d\mu = (1/q)\int^{\mu}f(\mu)d\mu,$$

where $f(\mu)$ and $F(x,\mu)$ are the functions used by Mott in Appendix II, ref. 8. Mott finds $f(\mu) = (q/2)\,[\Gamma(1-iq)/\Gamma(1+iq)]\,[(1-\mu)/2]^{iq-1}$. Therefore, $\int^{\mu}f(\mu)d\mu = f(\mu)\,[(1-\mu)/2]\,[2i/q] = 2iR/q = 2F_0$. Hence, the above sum equals $2F_0/q$, which leads immediately to $G_1^{F_0}$ (see equation (5)).

Abstract

Exact formulae for the scattering of Dirac electrons by a Coulomb field have been developed by Mott.[8] We have used these to calculate the angular distribution of the scattering of fast electrons in mercury, a typical heavy element. Curves are given for energies of 23 kv and upwards. Above 2 MEV, the angular distribution has an asymptotic character. It is of Rutherford type for very small angles, but the scattering at 60° is about twice as large as Rutherford scattering. The polarization has also been calculated, and is approximately the same as found by Mott.[8]

[17] See N. F. Mott, Proc. Roy. Soc. A 135, 456 (1932), Appendix I.

TABLE II

		$q = .6$	.65	.73	.8	.9	1.0	1.2	1.5	2.0
Phase	$\rho =$ 0.8117957	−25° 21′ 29″	−26° 28′ 05″	−27° 53′ 31″	−28° 47′ 53″	−29° 33′ 47″	−29° 44′ 54″	−28° 32′ 27″	−23° 19′ 11″	− 7° 18′ 24″
of	1.9128545	13 26 21	14 43 09	16 50 33	18 46 49	21 41 19	24 46 20	31 29 33	43 1 34	66 11 43
$\Gamma(\rho+iq)$	2.9426200	31 15 41	33 55 40	38 13 15	42 0 45	47 29 06	53 01 52	64 22 11	82 3 53	113 38 22
	3.9571470	42 55 45	46 32 15	52 19 39	57 24 36	64 42 05	72 02 06	86 49 53	109 24 53	148 16 39
	4.9657841								130 20 4	175 17 53
Phase	$k =$ 1	−15 37 38	−16 15 52	−17 01 33	−17 25 51	−17 35 39	−17 16 58	−15 18 59	− 9 20 9	7 25 42
of	2		16 45 34					34 52 41	46 58 27	70 51 47
$\Gamma(k+iq)$	3		34 45 49					65 50 31	83 50 39	115 51 47
	4		46 59 19					87 38 36	110 24 33	149 33 12
	5		56 13 07					104 20 33	130 57 55	176 7 6

TABLE III (A)

		$q = .6$		.65		.73		.8	
		R	I	R	I	R	I	R	I
$S\rho$	$\rho = 0.8117957$	− .661257	− .737616	− .641815	− .716033	− .616503	− .677422	− .599587	− .640552
	1.9128545	.437922	− .238834	.417370	− .266101	.380126	− .306680	.343585	− .338469
	2.9426200	− .147435	.298563	− .113585	.311789	− .057015	.324870	− .006262	.327871
	3.957147	.014093	− .249454	− .020387	− .248531	− .074289	− .237151	− .118326	− .217606
	4.9657841					.122447	.157169		
S_k	$k = 0$	− .864726	1.424790	− .827269	1.297108	− .767042	1.134977	− .714496	1.025668
	1	− .857482	− .004346	− .838412	.007243	− .807156	.029284	− .779154	.051726
	2	.324334	− .352372	.295867	− .372262	.247093	− .399443	.201618	− .418172
	3	− .082492	.316279	− .046733	.322405	.011228	.323688	.061610	.317191
	4	− .022504	− .246208	− .056379	− .240236	− .107837	− .221035	− .148329	− .195179
	5								
E_k	$k = 0$	− .314891	− .138379	− .309870	− .148273	− .303267	− .160234	− .299118	− .167165
	1	− .216201	.098431	− .224760	.088572	− .235459	.074248	− .242457	.063800
	2	.003989	− .049883	.003285	− .052190	+ .001880	− .055650	− .000375	− .059139
	3	.009326	.018572	.010878	.018813	.013799	.018952	.016307	.028268
	4	− .008299	− .007541	− .009453	− .007071	− .011107	− .006016	− .012648	− .004694
	5								

TABLE III (B)

.9		1.0		1.2		1.5		2.0	
R	I	R	I	R	I	R	I	R	I
− .582867	− .583947	− .572931	− .523943	− .565150	− .396262	− .554573	− .190285	− .436939	+ .154021
.285418	− .377227	.221361	− .406985	.081463	− .435295	− .130319	− .390192	− .345513	− .105764
.065610	.318281	.133449	.292783	.243148	.199747	+ .302736	− .004287	+ .090794	− .265992
− .172721	− .175749	− .213323	− .120503	− .241540	.011894	− .145885	+ .185890	+ .161289	+ .157648
						− .004433	− .192724	− .185642	+ .020740
− .640292	.908072	− .567349	.823478	− .424609	.717043	− .213428	+ .631580	+ .128199	+ .483286
− .738067	.087998	− .695413	.128064	− .603234	.214349	− .439256	+ .338743	− .090755	+ .437908
.133203	− .436070	.062158	− .442873	− .079751	− .421264	− .261465	− .302715	− .351120	+ .041411
.129561	.291806	.190357	.252516	.276939	.138167	+ .288965	− .073405	+ .022164	− .276463
− .195608	− .145691	− .227149	− .085010	− .234236	.049728	− .112145	+ .205470	+ .184659	+ .126101
						− .028744	− .189397	− .180150	+ .045043
− .295996	− .171937	− .296142	− .171185	− .304888	− .154350	− .328814	− .092081	− .326121	.100501
− .249303	.052495	− .253554	.045865	− .256918	.046468	− .251263	.078243	− .189467	.185228
− .004295	− .063598	− .009690	− .067749	− .023362	− .073336	− .052541	− .072104	− .104528	− .025771
.020617	.017562	.025459	.015216	.033066	.007836	.039373	− .011917	.015712	− .050990
− .014627	− .002140	− .016248	.001463	− .016372	.008983	− .008505	.021199	.022737	.020200
						− .004457	− .013611	− .018917	.003477

TABLE IV (A)

	$q=.6$		.65		.73		.8	
	M	Φ	M	Φ	M	Φ	M	Φ
$\Gamma(iq+1)$	.765451	− .272744	.734196	− .283868	.683876	− .297157	.640197	− .304226
$\Gamma(iq+\frac{1}{2})$	.965668	− .826353	.895450	− .856438	.792304	− .895082	.711081	− .920275
$\Gamma(iq-\frac{1}{2})$	1.236411	−3.191298	1.105411	−3.082928	.895445	−3.066419	.753745	−3.049671

(M is modulus—Φ is phase)

TABLE IV (B)

.9		1.0		1.2		1.5		2.0	
M	Φ	M	Φ	M	Φ	M	Φ	M	Φ
.579433	− .307074	.521564	− .301641	.417033	− .267320	.290985	− .162940	.153190	.129647
.608642	− .944008	.520591	− .955008	.380495	− .944631	.237569	− .863152	.108321	− .592538
.591165	−3.021903	.465631	−2.989452	.292688	−2.910218				

TABLE V (A)

		$\theta = 15°$		30°		45°		60°	
Phase of $(\sin\frac{1}{2}\theta)^{-2iq}$	$q=$ .6		2.443417		1.621951		1.152659		.831777
	.65		2.647036		1.757114		1.248711		.901091
	.73		2.972825		1.973374		1.402399		1.011995
	.8		3.257890		2.162602		1.536875		1.109036
	.9		3.665126		2.432927		1.728985		1.247665
	1.0		4.072363		2.703252		1.921094		1.386294
	1.2		4.886835		3.243903		2.305313		1.663553
	1.5								
	2.0								
		R	I	R	I	R	I	R	I
$2F_0/i$	$q=$ .6	−.321329	−.946968	.474445	−.880285	.821267	−.570544	.959299	−.282394
	.65	−.486870	−.873474	.372237	−.928138	.776959	−.629551	.944950	−.327215
	.73	−.722709	−.691143	.190563	−.981675	.690884	−.722965	.914032	−.405642
	.8	−.881317	−.472525	.016645	−.999861	.599097	−.800677	.877302	−.479939
	.9	−.995897	−.090491	−.245448	−.969410	.440325	−.897839	.805951	−.591983
	1.0	−.946854	.321664	−.504820	−.863225	.250295	−.968170	.708793	−.705417
	1.2	−.352457	.935828	−.907992	−.418988	−.198547	−.980091	.427643	−.903948
	1.5								
	2.0								
$2F_1/i$	$q=$ .6	.1446	.0346	.1345	.1647	.1036	.3435	.0407	.5308
	.65	.1443	.0151	.1392	.1468	.1136	.3283	.0532	.5188
	.73	.1386	−.0157	.1436	.1173	.1294	.3016	.0762	.4963
	.8	.1286	−.0410	.1444	.0924	.1428	.2778	.0987	.4750
	.9	.1068	−.0735	.1399	.0554	.1593	.2379	.1325	.4373
	1.0	.0777	−.0987	.1295	.0222	.1727	.1973	.1669	.3952
	1.2	.0062	−.1203	.0935	−.0295	.1846	.1139	.2269	.2962
	1.5								
	2.0								

TABLE V (B)

90°		120°		150°		180°	
	.415888		.172609		.041602		
	.450546		.186993		.045069		
	.505997		.210008		.050616		
	.554518		.230146		.055469		
	.623833		.258914		.062403		
	.693147		.287682		.069336		
	.831777		.345219		.083204		
	1.039721		.431523		.104005		
	1.386294		.575364		.138673		
R	I	R	I	R	I	R	. I
.991614	.129237	.931282	.364298	.875713	.482832	.854874	.518836
.993141	.116923	.928389	.371610	.866491	.499193	.843119	.537726
.996103	.088201	.927059	.374916	.855803	.517302	.828533	.559940
.998546	.053907	.929292	.369346	.850962	.525227	.820535	.571597
.999953	− .009684	.937565	.347810	.851611	.524175	.817265	.576261
.995965	− .089744	.950610	.310388	.860805	.508935	.823478	.567349
.956179	− .292782	.982113	.188292	.899821	.436260	.860452	.509531
.755852	− .654762	.994425	− .105448	.975487	.220059	.947370	.320142
− .074722	− .997204	.671431	− .741067	.921851	− .387545	.966571	− .256398
− .1800	.8202	− .4250	.9791	− .5673	1.0844	− .6035	1.1316
− .1753	.8135	− .4358	.9728	− .5854	1.0740	− .6220	1.1187
− .1599	.8012	− .4422	.9642	− .6016	1.0596	− .6381	1.0997
− .1394	.7880	− .4378	.9574	− .6040	1.0528	− .6397	1.0913
− .0989	.7705	− .4148	.9558	− .5910	1.0420	− .6271	1.0698
− .0486	.7482	− .3749	.9569	− .5591	1.0410	− .5956	1.0608
.0724	.6889	− .2478	.9641	− .4460	1.0549	− .4890	1.0591
.2721	.5808	.0349	.9140	− .1709	1.0787	− .2282	1.1665
.5000	.1506	.5550	.6104	.4669	.8976	.4725	.9627

TABLE VI

Function	q	$\theta = 15°$ R	$\theta = 15°$ I	30° R	30° I	45° R	45° I	60° R	60° I
$F(\frac{1}{2}, iq+1, iq+\frac{1}{2}; \sin^2 \frac{1}{2}\theta)$	$q =$ .6	1.01220	− .00428	1.05037	− .01791	1.11971	− .04358		
	.65	1.01187	− .00420	1.04900	− .01761	1.11636	− .04288	1.22389	− .08540
	.73	1.01142	− .00406	1.04709	− .01700	1.11171	− .04143	1.21417	− .08234
	.8	1.01108	− .00391	1.04569	− .01640	1.10831	− .04001	1.20787	− .07988
	.9	1.01069	− .00370	1.04405	− .01551	1.10430	− .03787	1.19986	− .07575
	1.0	1.01038	− .00348	1.04274	− .01463	1.10109	− .03576	1.19343	− .07165
	1.2	1.00993	− .00309	1.04083	− .01301	1.09640	− .03188	1.18398	− .06408

Function	q	$\theta = 15°$ R	$\theta = 15°$ I	30° R	30° I	45° R	45° I	60° R	60° I
$F(3/2, -iq+1, -iq+3/2; \sin^2 \frac{1}{2}\theta)$	$q =$ .6	1.01853	− .00301	1.07675	− .01269	1.18350	− .03128		
	.65	1.01870	− .00318	1.07750	− .01344	1.18527	− .03315	1.35978	− .06724
	.73	1.01899	− .00343	1.07871	− .01451	1.18816	− .03585	1.3646	− .07256
	.8	1.01925	− .00362	1.07980	− .01533	1.19082	− .03794	1.37073	− .07725
	.9	1.01963	− .00385	1.08137	− .01632	1.19464	− .04047	1.37828	− .08265
	1.0	1.02000	− .00403	1.08294	− .01710	1.19845	− .04251	1.38584	− .08706
	1.2	1.02072	− .00427	1.08596	− .01815	1.20585	− .04534	1.40068	− .09350

Function	q	$\theta = 15°$ R	$\theta = 15°$ I	30° R	30° I	45° R	45° I	60° R	60° I
$\lvert\Gamma(iq+1)\rvert^2 F(\frac{1}{2}, iq+1, 2; \cos^2 \frac{1}{2}\theta)$	$q =$ .6	.7095	.4385	.7860	.4146	.8058	.3087	.7490	.1536
	.65	.69276	.39466	.72420	.29958	.71174	.21419	.68282	.14880
	.73	.55753	.35519	.60153	.28196	.60321	.20689	.58587	.14696
	.8	.45668	.31610	.50619	.26071	.51679	.19477	.50737	.13951
	.9	.33924	.25888	.38940	.22630	.40853	.17392	.40761	.12644
	1.0	.24990	.20523	.29480	.19062	.31824	.15116	.32318	.11172
	1.2	.13556	.12009	.16282	.12565	.18614	.10692	.19660	.08208

Function	q	90° R	90° I	120° R	120° I	150° R	150° I
$F(\frac{1}{2}, iq+1, 2; \cos^2 \frac{1}{2}\theta)$	$q =$ .6	1.15909	.11999	1.06981	.04614	1.01722	.01058
	.65	1.15697	.12946	1.06946	.04997	1.01720	.01146
	.73	1.15321	.14560	1.06885	.05603	1.01717	.01287
	.8	1.14969	.15857	1.06826	.06144	1.01714	.01410
	.9	1.14403	.17771	1.06732	.06907	1.01708	.01587
	1.0	1.13773	.19661	1.06628	.07668	1.01703	.01763
	1.2	1.12326	.23360	1.06386	.09185	1.01689	.02115
	1.5	1.09616	.28577	1.05942	.11440	1.01665	.02644
	2.0	1.04129	.36577	1.04992	.15140	1.01611	.03523

TABLE VII (A)

		$\theta = 15°$		30°		45°		60°	
		R	I	R	I	R	I	R	I
G_0	$q =$ .6	− 5.561	−16.390	1.9824	−3.6782	1.4360	− .9976	.8633	− .2541
	.65	− 9.129	−16.378	1.6849	−4.2013	1.4717	−1.1925	.9213	− .3190
	.73	−15.219	−14.554	.9687	−4.9906	1.4697	−1.5380	1.0008	− .4441
	.8	−20.339	−10.905	.0927	−5.5705	1.3967	−1.8666	1.0527	− .5759
	.9	−25.856	− 2.349	−1.5383	−6.0759	1.1548	−2.3548	1.0880	− .7991
	1.0	−27.314	9.279	−3.5156	−6.0115	.7294	−2.8214	1.0631	−1.0581
	1.2	−12.201	32.395	−7.5880	−3.5014	− .6943	−3.4274	.7697	−1.6271
	1.5								
	2.0								
G_1^E	$q =$ .6	.2313	.0475	.2082	.0414	.1770	.0293	.1454	.0098
	.65	.2287	.0559	.2057	.0505	.1747	.0389	.1435	.0193
	.73	.2235	.0653	.2008	.0610	.1702	.0507	.1395	.0316
	.8	.2274	.0694	.2013	.0665	.1669	.0577	.1335	.0395
	.9	.2127	.0696	.1910	.0692	.1616	.0632	.1318	.0471
	1.0	.2062	.0639	.1858	.0665	.1578	.0642	.1287	.0509
	1.2	.1943	.0435	.1774	.0514	.1530	.0559	.1255	.0487
	1.5								
	2.0								
$G_1^{F_0}$	$q =$ .6	.1542	− .0761	.1607	.0610	.1157	.1271	.0697	.1571
	.65	.1387	− .1011	.1659	.0441	.1235	.1193	.0754	.1543
	.73	.1044	− .1360	.1709	.0143	.1360	.1043	.0860	.1483
	.8	.0656	− .1582	.1707	− .0140	.1466	.0886	.0966	.1414
	.9	.0004	− .1711	.1615	− .0564	.1596	.0615	.1130	.1285
	1.0	− .0676	− .1570	.1403	− .0977	.1684	.0295	.1298	.1113
	1.2	− .1635	− .0495	.0612	− .1595	.1648	− .0448	.1589	.0627
	1.5								
	2.0								

TABLE VII (B)

90°		120°		150°		180°	
R	I	R	I	R	I	R	I
.2974	.0387	.0931	.0364	.0188	.0104	0	0
.3227	.0380	.1005	.0402	.0202	.0116		
.3635	.0321	.1127	.0456	.0224	.0135		
.3994	.0215	.1239	.0492	.0244	.0150		
.4499	− .0043	.1406	.0521	.0275	.0169		
.4979	− .0448	.1584	.0517	.0309	.0182		
.5737	− .1756	.1964	.0376	.0387	.0187		
.5668	− .4910	.2486	− .0263	.0525	.0118		
− .0747	− .9972	.2238	− .2470	.0661	− .0278		
.1012	− .0480	.0827	− .1079	.0713	− .1434		
.1006	− .0406	.0844	− .1035	.0754	− .1403		
.0985	− .0309	.0855	− .0979	.0802	− .1363		
.0936	− .0241	.0864	− .0941	.0842	− .1336		
.0919	− .0171	.0824	− .0907	.0841	− .1315		
.0877	− .0125	.0779	− .0891	.0830	− .1311		
.0801	− .0098	.0651	− .0896	.0740	− .1347		
.0636	− .0165	.0416	− .0926	.0438	− .1458		
.0455	− .0305	− .0220	− .0865	− .0455	− .1433		
.0003	.1719	− .0411	.1669	− .0625	.1601		
.0007	.1717	− .0440	.1660	− .0670	.1581		
.0034	.1714	− .0467	.1650	− .0723	.1554		
.0076	.1711	− .0472	.1646	− .0751	.1539		
.0166	.1703	− .0452	.1650	− .0765	.1530		
.0287	.1685	− .0400	.1662	− .0751	.1536		
.0606	.1597	− .0210	.1695	− .0642	.1583		
.1184	.1229	.0269	.1685	− .0287	.1682		
.1695	− .0194	.1308	.1094	.0723	.1545		

TABLE VIII (A)

		$\theta = 15°$		30°		45°		60°	
		R	I	R	I	R	I	R	I
	$q =$ .6	1.4037	− 1.6269	.9867	− .1562	.5692	.1272	.3283	.1666
	.65	1.1913	− 1.8108	.9943	− .2271	.5884	.1094	.3401	.1659
G_1^{Φ}	.73	.7630	− 2.0428	.9880	− .3526	.6185	.0747	.3614	.1640
	.8	.3104	− 2.1584	.9577	− .4774	.6414	.0308	.3815	.1541
	.9	− .4114	− 2.1232	.8662	− .6659	.6646	− .0502	.4110	.1290
	1.0	− 1.1259	− 1.8065	.7111	− .8474	.6702	− .1505	.4391	.0904
	1.2	− 2.0026	− .3885	.2160	−1.0877	.6021	− .3876	.4772	− .0268
	1.5								
	2.0								
	$q =$ .6	.4561	− .0883	.3577	.3044	.1135	.4625	− .1242	.5000
	.65	.4291	− .1713	.3906	.2557	.1505	.4452	− .0957	.4991
F	.73	.3535	− .2921	.4321	.1671	.2106	.4102	− .0454	.4951
	.8	.2567	− .3763	.4537	.0805	.2614	.3709	.0024	.4880
	.9	.0820	− .4445	.4569	− .0527	.3299	.2998	.0773	.4692
	1.0	− .1114	− .4345	.4204	− .1876	.3854	.2115	.1550	.4378
	1.2	− .4077	− .1730	.2242	− .4072	.4330	− .0069	.3038	.3272
	1.5								
	2.0								
	$q =$ .6	− 3.772	−18.046	3.338	−3.732	2.2980	− .7138	1.4070	.0796
	.65	− 7.570	−18.234	3.051	−4.333	2.3584	− .9248	1.4804	.0205
G	.73	−14.128	−16.668	2.329	−5.268	2.395	−1.308	1.5879	− .1002
	.8	−19.735	−13.152	1.422	−5.995	2 3518	−1.6895	1.6645	− .2408
	.9	−26.054	− 4.574	− .319	−6.729	2.1408	−2.2802	1.7440	− .4944
	1.0	−28.301	7.379	−2.478	−6.890	1.7259	−2.8782	1.7609	− .8054
	1.2	−14.173	32.001	−7.133	−4.697	.2257	−3.8041	1.5315	−1.5424
	1.5								
	2.0								

TABLE VIII (B)

90°		120°		150°		180°	
R	I	R	I	R	I	R	I
.1095	.1097	.0328	.0479	.0064	.0116		
.1114	.1150	.0325	.0506	.0061	.0123		
.1158	.1222	.0323	.0544	.0058	.0132		
.1213	.1266	.0327	.0573	.0057	.0139		
.1313	.1302	.0343	.0607	.0057	.0148		
.1440	.1301	.0371	.0633	.0060	.0156		
.1759	.1165	.0464	.0653	.0075	.0167		
.2261	.0556	.0682	.0577	.0119	.0166		
.2183	− .1447	.1027	.0031	.0221	.0083		
− .4747	.4057	− .6717	.2531	− .7836	.1541	− .8252	.1256
− .4652	.4088	− .6722	.2462	− .7866	.1405	− .8282	.1105
− .4447	.4181	− .6696	.2424	− .7885	.1271	− .8298	.0951
− .4209	.4295	− .6633	.2457	− .7890	.1234	− .8314	.0903
− .3804	.4505	− .6518	.2613	− .7830	.1302	− .8230	.0950
− .3292	.4736	− .6336	.2878	− .7749	.1508	− .8140	.1139
− .1980	.5143	− .5762	.3671	− .7456	.2268	− .7843	.1857
.0369	.5140	− .4043	.5146	− .6494	.4022	− .7433	.3595
.4233	.2126	.0653	.6132	− .2550	.6943	− .3532	.7195
.5086	.2725	.1677	.1435	.0340	.0387		
.5356	.2842	.1735	.1534	.0347	.0417		
.5814	.2950	.1839	.1671	.0362	.0459		
.6220	.2952	.1959	.1771	.0392	.0493		
.6899	.2791	.2122	.1873	.0408	.0534		
.7586	.2413	.2334	.1921	.0448	.0565		
.8904	.0908	.2870	.1829	.0561	.0591		
.9750	− .3290	.3854	.1074	.0796	.0510		
.3586	−1.1919	.4355	− .2209	.1151	− .0083		